

Turing and Ordinal Logic

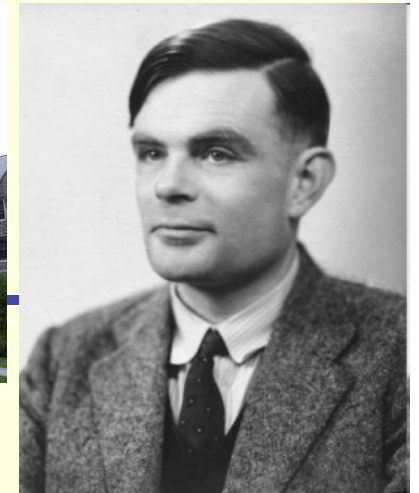
(Or "To Infinity and Beyond ...")

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Turing at Princeton



- Arrived in September 1936
 - Worked on type theory and group theory
 - Spent June-September 1937 back in England
 - Returned to Princeton in September 1937
 - Procter Fellow (on \$2000 per year) (!)
 - Church's PhD student (!!)
 - *Systems of Logic Based on Ordinals* completed in May, 1938 (!!!)
 - Returned to Cambridge in July, 1938
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"Overcoming" Gödel's Incompleteness

For any logic L_1 : A_1 is true but unprovable in L_1

L_2 is $L_1 + A_1$

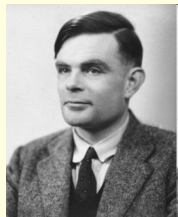
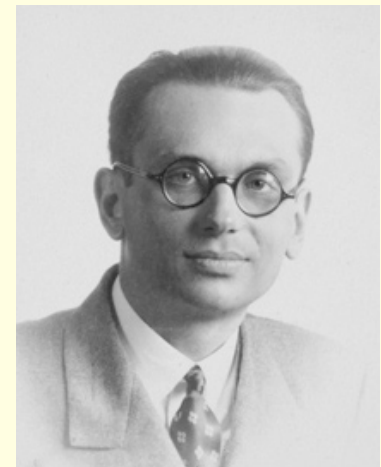
A_2 is true but unprovable in L_2

L_3 is $L_2 + A_2$

A_3 is true but unprovable in L_3

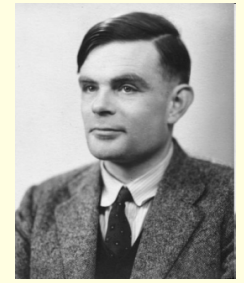
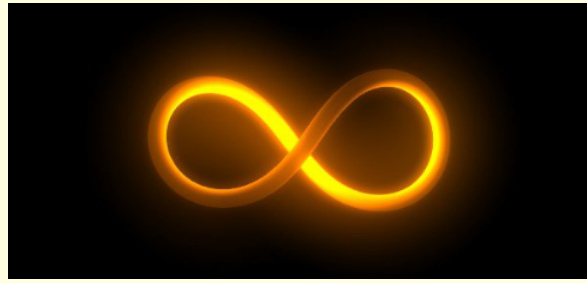
L_4 is $L_3 + A_3$

...



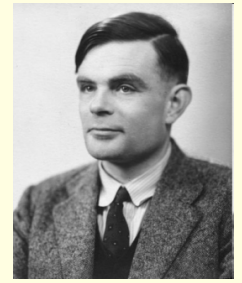
`` The situation is not quite so simple as is suggested by this crude argument'' (first page of 68)

Ordinal Logics



- In general, need to add an infinite number of axioms
- Create 'ordinal logics' in the same way as ordinal numbers
- Pass to the transfinite in order to obtain completeness
- (... lots of technicality omitted ...)
- Completeness obtained (for Π^0_1 statements)
- "Proofs" now not just 'mechanical'
- Need to recognise when a formula is an 'ordinal formula'
- This is at least as hard as Π^0_1 statements (!!)
- Trades 'mechanical proofs' for completeness
- Isolates where the 'non-mechanical' part is

Oracles

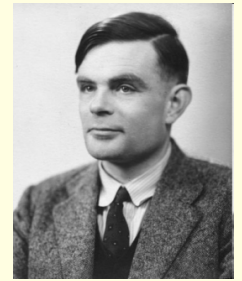
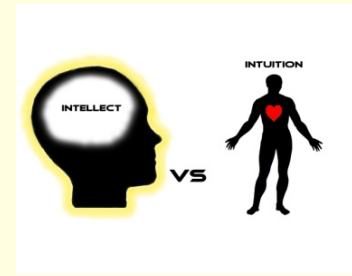


Oracle: Answer an unsolvable problem

o-machines: add this capability to Turing machines

- Provides computational conception of limit ordinal process
- Showed that the halting problem for o-machines cannot be solved by an o-machine
- Genesis of the idea of relative computability & generalised recursion theory

Formalising 'Intuition'



- Turing divided mathematical reasoning into **ingenuity** and **intuition**
- Ordinal logics formalise this distinction via ordinal notation or 'oracle'
- Extension of work on mechanising human reasoning
- Turing's most 'mainstream' work ??

